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# Porous-elastic system with boundary dissipation of fractional derivative type

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Abstract. This paper deals with the solution and asymptotic analysis for a porous-elastic system with two dynamic control boundary conditions of fractional derivative type. We consider an augmented model. The energy function is presented, and the dissipative property of the system is established. We use the semigroup theory. The existence and uniqueness of the solution are obtained by applying the well-known Lumer-Phillips Theorem. We present two results for the asymptotic behavior: Strong stability of the $C_0$-semigroup associated with the system using Arendt-Batty and Lyubich-Vũ's general criterion and polynomial stability applying Borichev-Tomilov's Theorem.

## 1. Introduction

This manuscript is concerned with the one-dimensional linear equations of a homogeneous and isotropic porous elastic solid with fractional dissipation. The theory of elastic solids with voids was established by Cowin and Nunziato [9, 10, 22] as an extension of classical elasticity theory that allows the treatment of porous solids with elastic materials that have good physical properties.

Denoting by $u$ and $\phi$ the displacement of the solid elastic material and the volume fraction, respectively. In the one-dimensional case, the evolution equations are

$$\begin{cases} \rho u_{tt} = T_x, \\ J\phi_{tt} = H_x + G, \end{cases} \tag{1}$$

where, $T$ is the stress tensor, $H$ is the equilibrated stress, $G$ is the equilibrated body force. $J = \rho k$ where $\rho$ is the mass density and $k$ is the equilibrated inertia, that are assumed positives.

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The constitutive equations are

$$
\begin{cases} T = \mu u_x + b\phi, \\ H = \delta\phi_x, \\ G = -bu_x - \xi\phi. \end{cases} \tag{2}
$$

The constitutive coefficients satisfy the conditions

$$\mu > 0, \;\; b > 0, \;\; \delta > 0, \;\; \xi > 0, \quad \text{and} \;\; b^2 \leqslant \mu\xi.$$

Since $b^2 \leqslant \mu\xi$ we have

$$
\mu|u_x|^2 + 2bu_x\phi + \xi|\phi|^2 = \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)^2 + \left(\mu - \frac{b^2}{\xi}\right)|u_x|^2 \geq 0. \tag{3}
$$

Introducing the constitutive equations (2) into the evolution equations (1), we get the system

$$
\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad \text{in } (0,L) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0 \quad \text{in } (0,L) \times (0,\infty). \end{cases} \tag{4}
$$

By using (3), the energy of the system is defined by

$$
E(t) = \frac{1}{2}\int_0^L \left[\rho|u_t|^2 + J|\phi_t|^2 + \delta|\phi_x|^2 + \mu|u_x|^2 + 2bu_x\phi + \xi|\phi|^2\right] dx.
$$

A direct calculation leads to $\frac{d}{dt}E(t) = 0$, and then, the system (4) is conservative. We must have a model for realistic situations where energy decreases in time, that is, $\frac{d}{dt}E(t) < 0$. For this purpose, we consider two dynamic controls of the fractional derivative type in the boundary conditions that act as dissipative mechanisms, given by

$$u(0,t) = \phi(0,t) = 0, \;\text{ in } (0,\infty), \tag{5}$$

$$u_x(L,t) = -\gamma_1 \partial_t^{\alpha,\eta} u(L,t) \text{ in } (0,+\infty), \; \eta \geqslant 0, \; 0 < \alpha < 1, \tag{6}$$

$$\phi_x(L,t) = -\gamma_2 \partial_t^{\alpha,\eta} \phi(L,t) \text{ in } (0,+\infty), \; \eta \geqslant 0, \; 0 < \alpha < 1, \tag{7}$$

where $\gamma_i > 0, \; i = 1, 2$. The notation $\partial_t^{\alpha,\eta}$ represents the generalized fractional derivative of the Caputo type of order $\alpha$, $0 < \alpha < 1$, $\eta \geq 0$.

We take initial data as

$$
\begin{cases} (u(x,0), \phi(x,0)) = (u_0(x), \phi_0(x)), \;\text{ in } (0,L), \\ (u_t(x,0), \phi_t(x,0)) = (u_1(x), \phi_1(x)), \;\text{ in } (0,L), \end{cases} \tag{8}
$$

where $(u_0, u_1, \phi_0, \phi_1)$ belong to a suitable functional space that will be defined later.

There are many definitions for fractional derivatives [11], among which Riemann-Liouville's and Caputo's definitions are most widely used [20]. A new definition of fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variable was given Caputo-Fabrizio [7]. The Caputo-Fabrizio fractional integral of order $\alpha$, $0<\alpha<1$, is defined by

$$I^{\alpha}w(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)ds,$$

where $\Gamma$ is the well-known gamma function, $w\in L^1(0,L)$ and $t>0$.

The Caputo fractional derivative operator of order $\alpha$ is defined by

$$D^{\alpha}w(t)=I^{1-\alpha}Dw(t):=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{dw}{ds}(s)ds,$$

with $w\in W^{1,1}(0,L)$ and $t>0$.

In this work, we consider the definition of fractional integro-differential operators with weight exponential, see Choi and MacCamy [8].

Let $0<\alpha<1$, $\eta\geq 0$. The exponential fractional integral of order $\alpha$ is defined by

$$\left[I^{\alpha,\,\eta}w\right](t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}w(\tau)d\tau,\ w\in L^1(0,L),\ t>0. \tag{9}$$

The exponential fractional derivative operator of order $\alpha$ is defined by

$$\begin{gathered}\partial_t^{\alpha,\eta}w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)ds,\\ w\in W^{1,1}(0,L),\ t>0.\end{gathered} \tag{10}$$

From (9) and (10), we deduce

$$\partial_t^{\alpha,\eta}w(t)=[I^{1-\alpha,\,\eta}w'](t). \tag{11}$$

The theory of porous elasticity is used in engineering, such as vehicles, airplanes, and space structures, and attracts the attention of researchers. See, for instance, [13, 14, 16, 25, 26].

Quintanilla [28] considered the following system

$$\begin{cases}\rho u_{tt}-\mu u_{xx}-b\phi_x=0 \text{ in } (0,L)\times(0,\infty),\\ J\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\tau\phi_t=0 \text{ in } (0,L)\times(0,\infty).\end{cases} \tag{12}$$

Note that in (12) the control is given by the frictional damping $\tau\phi_t, \tau>0$. He used Hurtwitz theorem to prove that the frictional damping through porousviscosity is not strong enough to obtain an exponential decay but only a slow decay.

Apalara [2] studied the following problem

$$\begin{cases}\rho u_{tt}-\mu u_{xx}-b\phi_x=0 \text{ in } (0,1)\times(0,\infty),\\ J\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi+\displaystyle\int_0^t g(t-s)\phi_{xx}(x,s)ds=0 \text{ in } (0,1)\times(0,\infty),\end{cases}$$

and proved that the unique dissipation given by the memory term is strong enough to exponentially stabilize the system, depending on the kernel $g$ of the memory term and the wave speeds of the system.

the porous-elastic system with microtemperature is considered in [3]. An one-dimensional porous-elastic system with thermoelasticity of type III was investigated in [19]. Asymptotic behavior for a porous-elastic system with fractional derivative-type internal dissipation was analized in [23]. Exponential stability for a porous elastic system with fractional damping and fractional-order time delay was studied in [24].

In Section 2 the augmented model is presented. In Section 3 we introduce the energy functional and establish the dissipativity property of the system. In Section 4 we obtain the existence and uniqueness theorem for the augmented model by using the semigroup theory of linear operators. In section 5, by using general criteria due to Arendt-Batty and Lyubich-Vũ, we prove the strong stability of the $C_0$-semigroup associated with the augmented model. In section 6, by using the Borichev-Tomilov Theorem, we show the polynomial stability.

## 2. Augmented model

**Proposition 1** (See [21]). *Let $\omega$ be a function*

$$\omega(y) = |y|^{\frac{2\alpha-1}{2}}, \qquad y \in (-\infty, +\infty), \;\; 0 < \alpha < 1.$$

*Then, the relation between the Input $\mathcal{U}$ and the Output $\mathcal{O}$ of the following system*

$$\begin{cases} \varphi_t(y,t) + y^2\varphi(y,t) + \eta\varphi(y,t) - \mathcal{U}(t)\omega(y) = 0, \;\; \eta \geqslant 0, \; t > 0, \\ \varphi(y,0) = 0, \\ \mathcal{O}(t) = \gamma \displaystyle\int_{-\infty}^{+\infty} \omega(y)\varphi(y,t)dy, \end{cases}$$

*where* $\gamma = \dfrac{\sin(\alpha\pi)}{\pi} = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ *and* $\mathcal{U} \in C([0, +\infty))$, *is given by*

$$\mathcal{O}(t) = I^{1-\alpha,\,\eta}\mathcal{U}(t) = D^{\alpha,\,\eta}\mathcal{U}(t).$$

The strategy is the elimination of the fractional derivatives in time. To achieve this, we exploit the technique from [17]. Applying Proposition 1 with $\mathcal{U}(t) = u_t(L,t)$ and taking into account (11), we deduce

$$\begin{aligned} \gamma \int_{-\infty}^{\infty} \omega(y)\varphi(y,t)\; dy &= \mathcal{O}(t) \\ &= I^{1-\alpha,\eta}\mathcal{U}(t) \\ &= I^{1-\alpha,\eta}u_t(L,t) \\ &= \partial_t^{\alpha,\eta}u(L,t). \end{aligned}$$

Now, by using

$$\gamma\int_{-\infty}^{\infty}\omega(y)\varphi(y,t)\ dy=\partial_t^{\alpha,\eta}u(L,t)$$

we reformulate system (4)–(8) into the augmented model with $\eta\geqslant 0$, $t>0$,

$$\begin{cases}\rho u_{tt}-\mu u_{xx}-b\phi_x=0 \quad \text{in} \quad (0,L)\times(0,\infty),\\ \varphi_{1t}(y,t)+[y^2+\eta]\varphi_1(y,t)-u_t(L,t)\omega(y)=0,\ \ y\in(-\infty,+\infty),\\ J\phi_{tt}-\delta\phi_{xx}+bu_x+\xi\phi=0 \quad \text{in} \quad (0,L)\times(0,\infty),\\ \varphi_{2t}(y,t)+[y^2+\eta]\varphi_2(y,t)-\phi_t(L,t)\omega(y)=0,\ \ y\in(-\infty,+\infty),\\ (u(x,0),\phi(x,0))=(u_0(x),\phi_0(x)), \quad \text{in} \quad (0,L),\\ (u_t(x,0),\phi_t(x,0))=(u_1(x),\phi_1(x)), \quad \text{in} \quad (0,L),\\ u(0,t)=\phi(0,t)=0, \quad \text{in} \quad (0,\infty),\\ u_x(L,t)=-C_1\displaystyle\int_{-\infty}^{+\infty}\omega(y)\varphi_1(y,t)dy, \quad C_1=\gamma_1\gamma,\\ \phi_x(L,t)=-C_2\displaystyle\int_{-\infty}^{+\infty}\omega(y)\varphi_2(y,t)dy, \quad C_2=\gamma_2\gamma,\\ \varphi_1(y,0)=\varphi_2(y,0)=0,\ \ y\in(-\infty,+\infty).\end{cases}\qquad(\mathbf{P})$$

## 3. Energy of the system

This section will show that the energy functional $E(t)$ associated with the augmented system ($\mathbf{P}$) is dissipative.

**Lemma 1.** *If* $\lambda\in D_\eta=\mathbb{C}\backslash]-\infty,-\eta]$ *then*

$$F_1(\lambda)=\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}d\xi=\frac{\pi}{\sin(\tau\pi)}(\lambda+\eta)^{\tau-1},$$

$$F_2(\lambda)=\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(\lambda+\eta+\xi^2)^2}d\xi=(1-\tau)\frac{\pi}{\sin(\tau\pi)}(\lambda+\eta)^{\tau-2},$$

$$F_n(\lambda)=\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{(\lambda+\eta+\xi^2)^n}d\xi=\frac{(1-\tau)(2-\tau)\cdots(n-1-\tau)}{(n-1)!}\frac{\pi}{\sin(\tau\pi)}(\lambda+\eta)^{\tau-n},$$

*Proof.* See [18], Lemma 2.1. □

**Lemma 2.** *If* $\lambda\in D=\mathbb{C}\backslash]-\infty,-\eta]$ *then*

$$\int_{-\infty}^{+\infty}\frac{\omega^2(y)}{\lambda+\eta+y^2}dy=\frac{\pi}{\sin(\alpha\pi)}(\lambda+\eta)^{\alpha-1}.$$

*Proof.* See [5], page 4. □

**Lemma 3.** *Let $0<\alpha<1$, $\eta\geqslant 0$, then the following integral*

$$I_2(\eta,\alpha)=\int_{-\infty}^{+\infty}\frac{|y|^{2\alpha-1}}{1+\eta+y^2}dy$$

*is well-defined.*

*Proof.* First, $I_2(\eta,\alpha)$ can be written as

$$I_2(\eta,\alpha)=2\int_0^1\frac{y^{2\alpha-1}}{1+\eta+y^2}dy+2\int_1^{+\infty}\frac{y^{2\alpha-1}}{1+\eta+y^2}dy.$$

We have

$$\frac{y^{2\alpha-1}}{1+\eta+y^2}\underset{0}{\sim}\frac{y^{2\alpha-1}}{1+\eta}\quad\text{and}\quad\frac{y^{2\alpha-1}}{1+\eta+y^2}\underset{+\infty}{\sim}\frac{1}{y^{3-2\alpha}}.$$

Since $0<\alpha<1$, then $I_2(\eta,\alpha)$ is well-defined. □

**Lemma 4.** *Let $0<\alpha<1$, $\eta\geqslant 0$, and $f_3(x,y)$, $f_6(x,y)\in L^2\Big((0,L)\times(-\infty,+\infty)\Big)$. Assume that ($\eta>0$ and $\lambda\in\mathbb{R}$) or ($\eta=0$ and $\lambda\in\mathbb{R}^*$), then the following integrals*

$$I_1(\lambda,\eta,\alpha)=i\lambda\gamma\int_{-\infty}^{+\infty}\frac{\omega^2(y)}{i\lambda+y^2+\eta}dy,\ I_2(\lambda,\eta,\alpha)=\gamma\int_{-\infty}^{+\infty}\frac{\omega^2(y)}{i\lambda+y^2+\eta}dy,$$

$$I_3(f_3,\lambda,\eta,\alpha)=\gamma\int_{-\infty}^{+\infty}\frac{\omega(y)f_3(x,y)}{i\lambda+y^2+\eta}dy,\ I_4(f_6,\lambda,\eta,\alpha)=\gamma\int_{-\infty}^{+\infty}\frac{\omega(y)f_6(x,y)}{i\lambda+y^2+\eta}dy,$$

*are well-defined.*

*Proof.* See [15], page 13. □

The energy $E(t)$ associated with the augmented system (**P**) is defined by

$$\begin{aligned}E(t)=&\frac{\rho}{2}\int_0^L|u_t|^2dx+\frac{J}{2}\int_0^L|\phi_t|^2dx\\&+\frac{\mu}{2}\int_0^L|u_x|^2dx+\frac{\delta}{2}\int_0^L|\phi_x|^2dx+b\int_0^L u_x\phi dx+\frac{\xi}{2}\int_0^L|\phi|^2dx\\&+\frac{\mu C_1}{2}\int_{-\infty}^{+\infty}|\varphi_1(y,t)|^2dy+\frac{\delta C_2}{2}\int_{-\infty}^{+\infty}|\varphi_2(y,t)|^2dy.\end{aligned}$$

**Proposition 2.** *The energy $E(t)$ satisfies*

$$\begin{aligned}\frac{d}{dt}E(t)=&-\mu C_1\int_{-\infty}^{+\infty}[y^2+\eta]|\varphi_1(y,t)|^2dy\\&-\delta C_2\int_{-\infty}^{+\infty}[y^2+\eta]|\varphi_2(y,t)|^2dy\leqslant 0.\end{aligned}\tag{13}$$

*Proof.* Multiplying $(\mathbf{P})_1$ by $u_t$, $(\mathbf{P})_3$ by $\phi_t$ and performing integration on $(0, L)$ we get

$$\frac{\rho}{2}\frac{d}{dt}\int_0^L |u_t|^2 dx - \mu\int_0^L u_{xx}u_t dx - b\int_0^L \phi_x u_t dx = 0, \tag{14}$$

$$\frac{J}{2}\frac{d}{dt}\int_0^L |\phi_t|^2 dx - \delta\int_0^L \phi_{xx}\phi_t dx + b\int_0^L u_x\phi_t dx + \xi\int_0^L \phi\phi_t dx = 0. \tag{15}$$

Adding (14) and (15), integrating by parts, and using the boundary conditions we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho|u_t|^2 + \mu|u_x|^2 + J|\phi_t|^2 + \delta|\phi_x|^2 + 2bu_x\phi + \xi|\phi|^2\right] dx \\
&= -\mu C_1 u_t(L,t)\int_{-\infty}^{+\infty} \omega(y)\varphi_1(y,t)dy \\
&\quad - \delta C_2\phi_t(L,t)\int_{-\infty}^{+\infty} \omega(y)\varphi_2(y,t)dy.
\end{aligned} \tag{16}$$

Multiplying $(\mathbf{P})_2$ by $\mu C_1\varphi_1$, $(\mathbf{P})_4$ by $\delta C_2\varphi_2$ respectively, and integrating on $(-\infty, +\infty)$, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} \mu C_1|\varphi_1(y,t)|^2 dy = &-\mu C_1\int_{-\infty}^{+\infty} [y^2+\eta]\varphi_1(y,t)\varphi_1 dy \\
&+\mu C_1\int_{-\infty}^{+\infty} u_t(L,t)\omega(y)\varphi_1 dy,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} \delta C_2|\varphi_2(y,t)|^2 dy = &-\delta C_2\int_{-\infty}^{+\infty} [y^2+\eta]\varphi_2(y,t)\varphi_2 dy \\
&+\delta C_2\int_{-\infty}^{+\infty} \phi_t(L,t)\omega(y)\varphi_2 dy.
\end{aligned} \tag{18}$$

Adding (17) and (18) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} \mu C_1|\varphi_1(y,t)|^2 dy + \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} \delta C_2|\varphi_2(y,t)|^2 dy \\
&= -\mu C_1\int_{-\infty}^{+\infty} [y^2+\eta]\varphi_1(y,t)\varphi_1 dy + \mu C_1\int_{-\infty}^{+\infty} u_t(L,t)\omega(y)\varphi_1 dy \\
&\quad -\delta C_2\int_{-\infty}^{+\infty} [y^2+\eta]\varphi_2(y,t)\varphi_2 dy + \delta C_2\int_{-\infty}^{+\infty} \phi_t(L,t)\omega(y)\varphi_2 dy.
\end{aligned} \tag{19}$$

Finally, adding (16)–(19) and using the energy $E(t)$, we obtain

$$\frac{d}{dt}E(t) = -\mu C_1 \int_{-\infty}^{+\infty} [y^2+\eta]\varphi_1(y,t)\varphi_1 dy \\ -\delta C_2 \int_{-\infty}^{+\infty} [y^2+\eta]\varphi_2(y,t)\varphi_2 dy. \qquad \square$$

## 4. Existence and uniqueness of solution

In this section, we study the well-posedness of the system (**P**) using the semigroup theory of linear operators.

4.1. **Semigroup set up.** Let $U = (u, v, \varphi_1, \phi, \psi, \varphi_2)^T$ where $v = u_t$ and $\psi = \phi_t$. The system (**P**) can be written as

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(0) = (u_0, u_1, \varphi_{1,0}, \phi_0, \psi_1, \varphi_{2,0})^T, \end{cases} \tag{20}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is given by

$$\mathcal{A}U := \begin{pmatrix} v \\ \frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x \\ -(y^2+\eta)\varphi_1 + v(L,t)\omega(y) \\ \psi \\ \frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi \\ -(y^2+\eta)\varphi_2 + \psi(L,t)\omega(y), \end{pmatrix} \tag{21}$$

with domain

$$D(\mathcal{A}) = \left\{ \mathcal{U} \in \mathcal{H} \;\middle|\; \begin{array}{l} u,\ \phi \in H^2(0,L) \cap H^1_L(0,L), \ \ v,\ \psi \in H^1_L(0,L), \\ -(y^2+\eta)\varphi_1 + v(L,t)\omega(y) \quad \in L^2(-\infty,+\infty), \\ -(y^2+\eta)\varphi_2 + \psi(L,t)\omega(y) \quad \in L^2(-\infty,+\infty), \\ u_x(L,t) + C_1 \int_{-\infty}^{+\infty} \omega(y)\varphi_1(y,t)dy = 0, \\ \phi_x(L,t) + C_2 \int_{-\infty}^{+\infty} \omega(y)\varphi_2(y,t)dy = 0 \\ |y|\varphi_1, \ \ |y|\varphi_2 \in L^2(-\infty,+\infty), \end{array} \right\}$$

and phase space

$$\mathcal{H} = \left(H^1_L(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty)\right)^2,$$

where $H^1_L(0,L) = \{\varphi \in H^1(0,L);\ \ \varphi(0) = 0\}$.

**Remark 1.** The condition $|y|\varphi_1, |y|\varphi_2 \in L^2(-\infty,+\infty)$ is imposed to insure the existence

$$\int_{-\infty}^{+\infty}[y^2+\eta]|\varphi_1(y)|^2dy \quad \text{and} \quad \int_{-\infty}^{+\infty}[y^2+\eta]|\varphi_2(y)|^2dy$$

in (13), respectively. In addition to the existence of

$$\int_{-\infty}^{+\infty}\omega(y)\varphi_1(y)dy \quad \text{and} \quad \int_{-\infty}^{+\infty}\omega(y)\varphi_2(y)dy$$

in $(\mathbf{P})_1$ and $(\mathbf{P})_3$, respectively (for more details, see Lemma B.1 in [1]).

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Let

$$U=(u,v,\varphi_1,\phi,\psi,\varphi_2)^T,\ \overline{U}=(\overline{u},\overline{v},\overline{\varphi_1},\overline{\phi},\overline{\psi},\overline{\varphi_2})^T\in\mathcal{H}.$$

The inner product in $\mathcal{H}$ is defined by

$$\langle U,\hat{U}\rangle_{\mathcal{H}}=\int_0^L\left[\rho v\hat{v}+J\psi\hat{\psi}+\mu u_x\hat{u}_x+\delta\phi_x\hat{\phi}_x+b(u_x\hat{\phi}+\hat{u}_x\phi)+\xi\phi\hat{\phi}\right]dx$$
$$+\mu C_1\int_{-\infty}^{+\infty}\varphi_1\hat{\varphi}_1dy+\delta C_2\int_{-\infty}^{+\infty}\varphi_2\hat{\varphi}_2dy.$$

We have

$$\langle U,U\rangle_{\mathcal{H}}=\int_0^L\left[\rho|v|^2+J|\psi|^2+\mu|u_x|^2+\delta|\phi_x|^2+2bu_x\phi+\xi|\phi|^2\right]dx$$
$$+\mu C_1\int_{-\infty}^{+\infty}|\varphi_1|^2dy+\delta C_2\int_{-\infty}^{+\infty}|\varphi_2|^2dy, \tag{22}$$

Note that

$$\mu\xi|u_x|^2+2b\xi u_x\phi+\xi^2|\phi|^2=(\mu\xi-b^2)|u_x|^2+|bu_x+\xi\phi|^2\geq 0, \tag{23}$$

therefore, (22) defines a norm on $\mathcal{H}$, that is,

$$||U||^2_{\mathcal{H}}=\langle U,U\rangle_{\mathcal{H}}.$$

Let $U=(u,v,\varphi_1,\phi,\psi,\varphi_2)^T\in D(\mathcal{A})$. A straight-right computation leads to

$$\langle \mathcal{A}U,\ U\rangle_{\mathcal{H}}=-\mu C_1\int_{-\infty}^{+\infty}(y^2+\eta)\varphi_1^2dy-\delta C_2\int_{-\infty}^{+\infty}(y^2+\eta)\varphi_2^2dy\leqslant 0, \tag{24}$$

and we conclude that $\mathcal{A}$ is a dissipative operator on $\mathcal{H}$.

4.2. **well-posed.** Our goal now is to solve the system (20). We use the Lumer-Phillips Theorem (See, Pazy [27]) to prove that $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $\mathcal{H}$.

**Theorem 1.** *The operator $\mathcal{A}$, defined in* (21)*, is the infinitesimal generator of a $C_0$-semigroup of contractions $\{S(t);\ t \geqslant 0\}$ on $\mathcal{H}$.*

*Proof.* Since $\mathcal{A}$ is densely defined and dissipative, it remains to prove that $\mathcal{A}$ is maximal. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, consider the resolvent equation

$$(I - \mathcal{A})U = F. \tag{25}$$

We need to show that the solution $U = (u, v, \varphi_1, \phi, \psi, \varphi_2)^T \in \mathcal{H}$ of (25) is in $D(\mathcal{A})$. The resolvent equation leads to

$$u - v = f_1 \ \in\ H^1_L(0, L), \tag{26}$$

$$v - \frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\phi_x = f_2 \ \in\ L^2(0, L), \tag{27}$$

$$\varphi_1 + (y^2 + \eta)\varphi_1 - v(L)\omega(y) = f_3 \ \in\ L^2\left(-\infty, +\infty\right), \tag{28}$$

$$\phi - \psi = f_4 \ \in\ H^1_L(0, L), \tag{29}$$

$$\psi - \frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi = f_5 \ \in\ L^2(0, L), \tag{30}$$

$$\varphi_2 + (y^2 + \eta)\varphi_2 - \psi(L)\omega(y) = f_6 \ \in\ L^2\left(-\infty, +\infty\right). \tag{31}$$

From (26) and (29) we get

$$\begin{cases} v = u - f_1, \\ \psi = \phi - f_4. \end{cases} \tag{32}$$

Of course $v, \psi \in H^1_0(0, L)$. Besides this, by (28) and (31), we can find $\varphi_i$ $(i = 1, 2)$ as

$$\begin{cases} \varphi_1 = \dfrac{f_3(y) + v(L)\omega(y)}{y^2 + \eta + 1}, \\[2ex] \varphi_2 = \dfrac{f_6(y) + \psi(L)\omega(y)}{y^2 + \eta + 1}. \end{cases} \tag{33}$$

Replacing (32) in (33), we obtain

$$\begin{cases} \varphi_1 = \dfrac{u(L)\omega(y)}{y^2 + \eta + 1} + \dfrac{f_3(y) - f_1(L)\omega(y)}{y^2 + \eta + 1}, \\[2ex] \varphi_2 = \dfrac{\phi(L)\omega(y)}{y^2 + \eta + 1} + \dfrac{f_6(y) - f_4(L)\omega(y)}{y^2 + \eta + 1}. \end{cases} \tag{34}$$

Using (32) in (27) and (30), the functions $u$ and $\phi$ satisfy the following system

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x = \rho(f_1 + f_2), \\ J\phi - \delta\phi_{xx} + bu_x + \xi\phi = J(f_4 + f_5). \end{cases} \tag{35}$$

Solve the systems (35) is equivalent to finding

$$(u,\phi)\in\left(H^2(0,L)\cap H^1_L(0,L)\right)^2,$$

such that

$$(36)\quad \begin{cases} \rho\int_0^L u\chi dx-\mu\int_0^L u_{xx}\chi dx-b\int_0^L \phi_x\chi dx=\rho\int_0^L (f_1+f_2)\chi dx,\\ J\int_0^L \phi\zeta dx-\delta\int_0^L \phi_{xx}\zeta dx+\\ \quad +b\int_0^L u_x\zeta dx+\xi\int_0^L \phi\zeta dx=J\int_0^L (f_4+f_5)\zeta dx, \end{cases}$$

for all $(\chi,\zeta)\in H^1_L(0,L)\times H^1_L(0,L)$. Performing by integration by parts, using the boundary conditions and using (34), it follows from (36) that the functions $u$ and $\phi$ satisfy the following system

$$(37)\quad \begin{cases} \rho\int_0^L u\chi dx+\tilde{C}_1u(L)\chi(L)+\mu\int_0^L u_x\chi_x dx+b\int_0^L \phi\chi_x dx=\\ \rho\int_0^L (f_1+f_2)\chi dx-\mu C_1\chi(L)\int_{-\infty}^{+\infty}\frac{\omega(y)}{y^2+\eta+1}f_3(y)dy+\tilde{C}_1f_1(L)\chi(L)\\ J\int_0^L \phi\zeta dx+\tilde{C}_2\phi(L)\zeta(L)+\delta\int_0^L \phi_x\zeta_x dx+b\int_0^L u_x\zeta dx+\xi\int_0^L \phi\zeta dx=\\ J\int_0^L (f_4+f_5)\zeta dx-\delta C_2\zeta(L)\int_{-\infty}^{+\infty}\frac{\omega(y)}{y^2+\eta+1}f_6(y)dy+\tilde{C}_2f_4(L)\zeta(L). \end{cases}$$

where

$$\tilde{C}_1=\mu C_1\int_{-\infty}^{+\infty}\frac{\omega^2(y)}{y^2+\eta+1}dy,\quad \text{and}\quad \tilde{C}_2=\delta C_2\int_{-\infty}^{+\infty}\frac{\omega^2(y)}{y^2+\eta+1}dy.$$

By (23) we have that

$$\mu\xi|u_x|^2+2b\xi u_x\phi+\xi^2|\phi|^2\geq 0.$$

We consider $H^1_0(0,L)\times H^1_0(0,L)$ with the norm

$$\begin{aligned}\|(u,\phi)\|^2=&\ \xi\rho\int_0^L|u|^2dx+\xi\mu\int_0^L|u_x|^2dx+2\xi b\int_0^L\phi u_x dx\\ &+\xi J\int_0^L|\phi|^2dx+\xi\delta\int_0^L|\phi_x|^2dx+\xi^2\int_0^L|\phi|^2dx\\ &+\xi\tilde{C}_1|u(L)|^2+\xi\tilde{C}_2|\phi(L)|^2.\end{aligned}$$

Adding the equations of the system (37), we construct a variational problem

$$B\big((u,\phi),\ (\chi,\zeta)\big) = L((\chi,\zeta)), \quad \forall(\chi,\zeta)\in H^1_L(0,L)\times H^1_L(0,L), \tag{38}$$

where

$$B:\ [H^1_L(0,L)\times H^1_L(0,L)]^2 \longrightarrow \mathbb{C}$$

is a sesquilinear, continuous and coercive form, and

$$L: H^1_L(0,L)\times H^1_L(0,L)\longrightarrow \mathbb{C}$$

linear and continuous, given by

$$\begin{aligned} B\big((u,\phi),\ (\chi,\zeta)\big) &= \xi\rho\int_0^L u\chi dx + \xi\mu\int_0^L u_x\chi_x dx + \xi b\int_0^L \phi\chi_x dx \\ &\quad + \xi J\int_0^L \phi\zeta dx + \xi\delta\int_0^L \phi_x\zeta_x dx + \xi b\int_0^L u_x\zeta dx \\ &\quad + \xi^2\int_0^L \phi\zeta dx + \xi\tilde{C_1}u(L)\chi(L) + \xi\tilde{C_2}\phi(L)\zeta(L) \end{aligned}$$

and

$$\begin{aligned} L((\chi,\zeta)) &= \xi\rho\int_0^L (f_1+f_2)\chi dx - \mu C_1\chi(L)\int_{-\infty}^{+\infty}\frac{\omega(y)}{y^2+\eta+1}f_3(y)dy \\ &\quad + \xi\tilde{C_1}f_1(L)\chi(L) + \xi J\int_0^L (f_4+f_5)\zeta dx \\ &\quad - \xi\delta C_2\zeta(L)\int_{-\infty}^{+\infty}\frac{\omega(y)}{y^2+\eta+1}f_6(y)dy + \xi\tilde{C_2}f_4(L)\zeta(L). \end{aligned}$$

Thus, applying the Lax-Milgram Lemma, we obtain the existence and uniqueness of solution $(u,\phi)\in H^1_L(0,L)\times H^1_L(0,L)$ of the problem (38), for all $(\chi,\zeta)\in H^1_0(0,L)\times H^1_0(0,L)$. From (32), we get $v,\ \psi\in H^1_0(0,L)$. In order to complete the existence of $U\in D(\mathcal{A})$, we need to prove $\varphi_i$ and $|y|\varphi_i\in L^2(-\infty,+\infty)$, $i=1,2$.

To do this, using (26), (28) and the fact that $\eta\geqslant 0$, we get

$$\varphi_1(y) = \frac{f_3(y)}{1+y^2+\eta} + \frac{u(L)|y|^{\frac{2\alpha-1}{2}}}{1+y^2+\eta} - \frac{f_1(L)|y|^{\frac{2\alpha-1}{2}}}{1+y^2+\eta}. \tag{39}$$

From (39), we get

$$\begin{aligned} \int_{-\infty}^{+\infty}|\varphi_1(y)|^2 dy &\leqslant 3\int_{-\infty}^{+\infty}\frac{|f_3(y)|^2}{(1+y^2+\eta)^2}dy \\ &\quad + 3\big(|u(L)|^2+|f_1(L)|^2\big)\int_{-\infty}^{+\infty}\frac{|y|^{2\alpha-1}}{(1+y^2+\eta)^2}dy. \end{aligned}$$

Using Lemma 3, it easy to observe that

$$\int_{-\infty}^{+\infty} \frac{|y|^{2\alpha-1}}{(1+y^2+\eta)^2}dy \leqslant \int_{-\infty}^{+\infty} \frac{|y|^{2\alpha-1}}{1+y^2+\eta}dy \leq \ I_2(\eta,\alpha) \leqslant +\infty,$$

that is,

$$\begin{aligned}\left(|u(L)|^2+|f_1(L)|^2\right)\int_{-\infty}^{+\infty} \frac{|y|^{2\alpha-1}}{(1+y^2+\eta)^2}dy \\ \leqslant I_2(\eta,\alpha)\left(|u(L)|^2+|f_1(L)|^2\right) < \infty.\end{aligned}$$

On the other hand, using the fact that $f_3 \in L^2\,(-\infty,+\infty)$, we obtain

$$\int_{-\infty}^{+\infty} \frac{|f_3(y)|^2}{(1+y^2+\eta)^2}dy \leqslant \frac{1}{(1+\eta)^2}\int_{-\infty}^{+\infty} |f_3(y)|^2 dy < \infty.$$

It follows that $\varphi_1 \in L^2\,(-\infty,+\infty)$. Next, using (39), we get

$$\begin{aligned}\int_{-\infty}^{+\infty} |y\varphi_1|^2 dy \leqslant 3\int_{-\infty}^{+\infty} \frac{y^2|f_3(y)|^2}{(1+y^2+\eta)^2}dy \\ + 6\left(|u(L)|^2+|f_1(L)|^2\right)\int_{0}^{+\infty} \frac{|y|^{2\alpha+1}}{(1+y^2+\eta)^2}dy.\end{aligned}$$

Since

$$\frac{y^{2\alpha+1}}{(1+\eta+y^2)^2} \underset{0}{\sim} \frac{y^{2\alpha+1}}{(1+\eta)^2} \quad \text{and} \quad \frac{y^{2\alpha+1}}{(1+\eta+y^2)^2} \underset{+\infty}{\sim} \frac{1}{y^{3-2\alpha}}.$$

Using the fact that $0<\alpha<1$, we obtain $\displaystyle\int_{0}^{+\infty} \frac{|y|^{2\alpha+1}}{(1+y^2+\eta)^2}dy$ is well-defined and consequently so is $\displaystyle\left(|u(L)|^2+|f_1(L)|^2\right)\int_{0}^{+\infty} \frac{|y|^{2\alpha+1}}{(1+y^2+\eta)^2}dy$. Now, using the fact that $f_3 \in L^2\,(-\infty,+\infty)$ and

$$\max_{y\in(-\infty,+\infty)} \frac{y^2}{(1+y^2+\eta)^2} = \frac{1}{(1+\eta)} < 1,$$

we get

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{y^2|f_3(y)|^2}{(1+y^2+\eta)^2}dy \leqslant \max_{y\in(-\infty,+\infty)} \frac{y^2}{(1+y^2+\eta)^2}\int_{-\infty}^{+\infty} |f_3(y)|^2 dy \\ < \int_{-\infty}^{+\infty} |f_3(y)|^2 dy < +\infty.\end{aligned}$$

It follows that $|y|\varphi_1 \in L^2\,(-\infty,+\infty)$. Similarly, it is proved that $\varphi_2,\ |y|\varphi_2 \in L^2\,(-\infty,+\infty)$. In this way, we prove that $\mathcal{A}$ is maximal. Consequently, by the Lumer-Phillips theorem, $\mathcal{A}$ is the infinitesimal generator of $C_0$-semigroup of contractions on $\mathcal{H}$. □

The existence and uniqueness of solution is given by the following theorem.

**Theorem 2.** *Defining $U(t) = e^{\mathcal{A}t}U(0)$, by general theory of semigroups of linear operators, we have*

(a) *If $U(0) \in \mathcal{H}$, then the system* (20) *has a unique solution*

$$U \in C\left(\mathbb{R}_0^+,\ \mathcal{H}\right).$$

(b) *If $U(0) \in \mathcal{D}(\mathcal{A})$, then the system* (20) *has a unique strong solution*

$$U \in C\left(\mathbb{R}_0^+,\ \mathcal{D}(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_0^+,\ \mathcal{H}\right).$$

## 5. Strong stability

In this section, we use a general Arendt-Batty [4] and Lyubich-Vũ [5] criterion to show the strong stability of the $C_0$-semigroup $S(t) = e^{t\mathcal{A}}$ associated with the system (**P**).

**Theorem 3.** *(Stability Theorem: [4], page 837) Let $\mathcal{A}$ be the generator of a bounded $C_0$-semigroup $\{S(t)\}_{t\geqslant 0}$ over a Hilbert space $\mathcal{H}$. If no eigenvalues of $\mathcal{A}$ lies on the imaginary axis $i\mathbb{R}$ and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, then $\{S(t)\}_{t\geqslant 0}$ is asymptotically stable. That is, $\lim\limits_{t\to\infty} \|S(t)x\|_{\mathcal{H}} = 0$ for all $x \in \mathcal{H}$.*

**Lemma 5** (Lax–Milgram–Fredholm, see [12])**.** *Let $V$ and $H$ be Hilbert spaces such that the embedding $V \subset H$ is compact and dense. Suppose that $a_V : V \times V \longrightarrow \mathbb{C}$ and $a_H : H \times H \longrightarrow \mathbb{C}$ are two bounded sesquilinear forms such that $a_V$ is $V$-coercive and $G : V \longrightarrow \mathbb{C}$ is a continuous conjugate linear form. The equation*

$$a_H(u, v) + a_V(u, v) = G(v), \quad \forall v \in V$$

*has either a unique solution $u \in V$ for all $G \in V'$ or has a nontrivial solution for $G = 0$.*

The main result in this section is the following theorem.

**Theorem 4.** *The $C_0$-semigroup $S(t) = e^{t\mathcal{A}}$ is asymptotically stable on $\mathcal{H}$.*

For the proof of Theorem 4, according to Theorem 3 of Arendt and Batty, we need to prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$. The argument for Theorem 4 relies on the subsequent lemmas.

**Lemma 6.** *Assume that $\eta \geqslant 0$. Then, ones has*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

*Proof.* We will use argument of contradiction. If $\mathcal{A}$ has a eigenvalue on the imaginary axis $i\mathbb{R}$, then there exists a $\lambda \neq 0$ and $U = (u, v, \varphi_1, \phi, \psi, \varphi_2)^T \in \mathcal{H}$ such that

$$\mathcal{A}U = i\lambda U, \quad U \neq 0. \tag{40}$$

Without loss of generality, we can assume $||U||_{\mathcal{H}} = 1$. The resolvent equation (40) leads to

$$i\lambda u - v = 0, \tag{41}$$

$$i\lambda v - \frac{\mu}{\rho} u_{xx} - \frac{b}{\rho}\phi_x = 0, \tag{42}$$

$$i\lambda\varphi_1 + (y^2+\eta)\varphi_1 - v(L)\omega(y) = 0, \tag{43}$$

$$i\lambda\phi - \psi = 0, \tag{44}$$

$$i\lambda\psi - \frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi = 0, \tag{45}$$

$$i\lambda\varphi_2 + (y^2+\eta)\varphi_2 - \psi(L)\omega(y) = 0. \tag{46}$$

Taking the real part in (40) we get $-Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = 0$. Then, by using (24) we obtain

$$\mu C_1 \int_{-\infty}^{+\infty} (y^2+\eta)|\varphi_1|^2 dy + \delta C_2 \int_{-\infty}^{+\infty} (y^2+\eta)|\varphi_2|^2 dy,$$

from where follows

$$\varphi_i \equiv 0, \quad i = 1, 2. \tag{47}$$

Then, from (43) and (46), it follows that

$$v(L) = \psi(L) = 0.$$

From (41), (44) and using (47) under boundary conditions $(\mathbf{P})_{7-8}$, we have

$$u(L) = \phi(L) = 0 \quad \text{and} \quad u_x(L) = \phi_x(L) = 0. \tag{48}$$

It follows from (41)–(42) and (44)–(45) that:

$$\begin{cases} -\lambda^2\rho u - \mu u_{xx} - b\phi_x = 0, \\ -\lambda^2 J\phi - \delta\phi_{xx} + bu_x + \xi\phi = 0. \end{cases} \tag{49}$$

Considering $X = (u, \phi, u_x, \phi_x)$, we can rewrite (48) and (49) as the following initial value problem

$$\begin{cases} \dfrac{dX}{dx} = \mathcal{A}X, \\ X(L) = 0, \end{cases} \tag{50}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{\lambda^2\rho}{\mu} & 0 & 0 & -\dfrac{b}{\mu} \\ 0 & \dfrac{-\lambda^2 J + \xi}{\delta} & \dfrac{b}{\delta} & 0 \end{pmatrix}.$$

By Picard's theorem, the ordinary differential equations of the system (50) has only one solution $X = 0$. Therefore, $u = 0,\ \ \phi = 0 \ \ \Rightarrow \ \ v = \psi = 0$, i.e., $U = 0$. □

**Lemma 7.** *Assume that $\eta = 0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0 \in \sigma(\mathcal{A})$.*

*Proof.* Set $F = (\sin(x), 0, 0, 0, 0, 0) \in \mathcal{H}$ and assume that there exists $U = (u, v, \varphi_1, \phi, \psi, \varphi_2) \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$, it follows that

$$v = -sin(x) \ \text{ in } \ (0, L) \ \ \text{ and } \ y^2\varphi_1 - \sin(L)|y|^{\frac{2\alpha-1}{2}} = 0.$$

From the above equation, we deduce that

$$\varphi_1(y) = \sin(L)|y|^{\frac{2\alpha-5}{2}} \neq L^2(-\infty, +\infty),$$

therefore the assumption of the existence of $U$ is false and consequently, the operator $-\mathcal{A}$ is not invertible. The proof is thus complete. □

**Lemma 8.** *Assume that ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$). Then, we have $\mathcal{R}(i\lambda I - \mathcal{A}) = \mathcal{H}$, where $\mathcal{R}(i\lambda I - \mathcal{A})$ denotes the Range of $i\lambda I - \mathcal{A}$.*

*Proof.* For $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, let $U = (u, v, \varphi_1, \phi, \psi, \varphi_2)^T \in D(\mathcal{A})$ be the solution of

$$i\lambda U - \mathcal{A}U = F.$$

Equivalently, we have

$$\tag{51} \begin{cases} i\lambda u - v = f_1, \\ i\lambda\rho v - \mu u_{xx} - b\phi_x = \rho f_2, \\ i\lambda\varphi_1 + (y^2 + \eta)\varphi_1 - v(L)\omega(y) = f_3, \\ i\lambda\phi - \psi = f_4, \\ i\lambda J\psi - \delta\phi_{xx} + bu_x + \xi\phi = Jf_5, \\ i\lambda\varphi_2 + (y^2 + \eta)\varphi_2 - \psi(L)\omega(y) = f_6. \end{cases}$$

It follows from $(51)_{3,6}$ that

$$\tag{52} \begin{cases} \varphi_1 = \dfrac{f_3(y)}{y^2 + \eta + i\lambda} - \dfrac{f_1(L)\omega(y)}{y^2 + \eta + i\lambda} + \dfrac{i\lambda u(L)\omega(y)}{y^2 + \eta + i\lambda}, \\ \varphi_2 = \dfrac{f_6(y)}{y^2 + \eta + i\lambda} - \dfrac{f_4(L)\omega(y)}{y^2 + \eta + i\lambda} + \dfrac{i\lambda\phi(L)\omega(y)}{y^2 + \eta + i\lambda}. \end{cases}$$

From $(51)_1 - (51)_2$ and $(51)_4 - (51)_5$, we obtain

$$\tag{53} \begin{cases} -\rho\lambda^2 u - \mu u_{xx} - b\phi_x = \rho(i\lambda f_1 + f_2), \\ -J\lambda^2\phi - \delta\phi_{xx} + bu_x + \xi\phi = J(i\lambda f_4 + f_5). \end{cases}$$

Solving the system (53) is equivalent to finding

$$(u,\ \phi) \in \left(H^2(0,L) \cap H^1_L(0,L)\right)^2,$$

such that

$$\begin{aligned} -\rho\lambda^2 \int_0^L u\chi dx - \mu \int_0^L u_{xx}\chi dx - b\int_0^L \phi_x \chi dx = \rho \int_0^L (i\lambda f_1 + f_2)\chi dx, \\ -J\lambda^2 \int_0^L \phi\zeta dx - \delta \int_0^L \phi_{xx}\zeta dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx = J\int_0^L (i\lambda f_4 + f_5)\zeta dx, \end{aligned} \tag{54}$$

for all $(\chi,\ \zeta) \in H^1_L(0,L) \times H^1_L(0,L)$. Using integration by parts and using (52) in (54), the functions $u$ and $\phi$ satisfy the following system

$$\begin{aligned} &-\rho\lambda^2 \int_0^L u\chi dx + \tilde{C}_1(\lambda,\eta,\alpha)\chi(L)v(L) + \mu\int_0^L u_x\chi_x dx + b\int_0^L \phi\chi_x dx \\ &= \rho\int_0^L (i\lambda f_1 + f_2)\chi dx - \mu C_1\chi(L)\int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_3(y)dy, \\ &-J\lambda^2 \int_0^L \phi\zeta dx + \tilde{C}_2(\lambda,\eta,\alpha)\zeta(L)\psi(L) + \delta\int_0^L \phi_x\zeta_x dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx \\ &= J\int_0^L (i\lambda f_4 + f_5)\zeta dx - \delta C_2\zeta(L)\int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_6(y)dy, \end{aligned} \tag{55}$$

where

$$\tilde{C}_1(\lambda,\eta,\alpha) = \mu C_1 \int_{-\infty}^{+\infty} \frac{\omega^2(y)}{y^2+\eta+i\lambda} dy,$$
$$\tilde{C}_2(\lambda,\eta,\alpha) = \delta C_2 \int_{-\infty}^{+\infty} \frac{\omega^2(y)}{y^2+\eta+i\lambda} dy.$$

Using again the $(51)_1 - (51)_4$, we deduce that

$$v(L) = i\lambda u(L) - f_1(L),$$
$$\psi(L) = i\lambda\phi(L) - f_4(L).$$

Substituting these equations, respectively, into equations (55), one has

$$\begin{aligned} &-\rho\lambda^2 \int_0^L u\chi dx + i\lambda\tilde{C}_1(\lambda,\eta,\alpha)u(L)\chi(L) + \mu\int_0^L u_x\chi_x dx + b\int_0^L \phi\chi_x dx \\ &= \rho\int_0^L (i\lambda f_1 + f_2)\chi dx - \mu C_1\chi(L)\int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_3(y)dy + \tilde{C}_1(\lambda,\eta,\alpha)f_1(L)\chi(L), \\ &-J\lambda^2 \int_0^L \phi\zeta dx + i\lambda\tilde{C}_2(\lambda,\eta,\alpha)\phi(L)\zeta(L) + \delta\int_0^L \phi_x\zeta_x dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx \\ &= J\int_0^L (i\lambda f_4 + f_5)\zeta dx - \delta C_2\zeta(L)\int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_6(y)dy + \tilde{C}_2(\lambda,\eta,\alpha)f_4(L)\zeta(L). \end{aligned} \tag{56}$$

As $\alpha \in (0,1)$ and $f_3,\ f_6 \in L^2(-\infty,+\infty)$, under the assumptions of the Lemma, it follows Lema 4 that the improper integrals in (56) above are well defined.

To solve (56), we distinguish two cases.

**Case 1.** $\eta > 0$ and $\lambda = 0$: System (56) becomes

$$
(57)\quad \begin{cases}
\mu \int_0^L u_x \chi_x dx + b \int_0^L \phi \chi_x dx = \\
\rho \int_0^L f_2 \chi dx - \mu C_1 \chi(L) \int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta} f_3(y) dy + \tilde{C}_1(0,\eta,\alpha) f_1(L)\chi(L), \\
\delta \int_0^L \phi_x \zeta_x dx + b \int_0^L u_x \zeta dx + \xi \int_0^L \phi \zeta dx = \\
J \int_0^L f_5 \zeta dx - \delta C_2 \zeta(L) \int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta} f_6(y) dy + \tilde{C}_2(0,\eta,\alpha) f_4(L)\zeta(L).
\end{cases}
$$

The left hand side of (57) is a bilinear continuous coercive form on $(H_L^1(0,L) \times H_L^1(0,L))^2$, and the right hand side of (57) is a linear continuous form on $H_L^1(0,L) \times H_L^1(0,L)$. Using Lax–Milgram theorem, we deduce that there exists a unique solution $(u,\phi) \in H_L^1(0,L) \times H_L^1(0,L)$ of the variational Problem (57). Hence, by applying the classical elliptic regularity we deduce that System (53) has a unique strong solution $(u,\phi) \in (H^2(0,L) \times H_L^1(0,L))^2$.

**Case 2.** $\eta \geqslant 0$ and $\lambda \in \mathbb{R}^*$: Note that we can rewrite (56) as

$$
(58)\qquad L_\lambda(U,V) + a_{(H_L^1(0,L))^2}(U,V) = l(V),
$$

where the sesquilinear forms

$$
L_\lambda : [L^2(0,L) \times L^2(0,L)]^2 \longrightarrow \mathbb{C},
$$

$$
a_{(H_L^1(0,L))^2} : [H_L^1(0,L) \times H_L^1(0,L)]^2 \longrightarrow \mathbb{C}
$$

and the antilinear form $l : H_L^1(0,L) \times H_L^1(0,L) \longrightarrow \mathbb{C}$ are defined by

$$
L_\lambda(U,V)_{H_{\mathbb{R}}^1} = \lambda^2 \int_0^L (\rho u \chi + \phi \zeta) dx - i\lambda \tilde{C}_1 u(L)\chi(L) - i\lambda \tilde{C}_2 \phi(L)\zeta(L),
$$

$$
a_{(H_0^1(0,L))^2}(U,V) = \mu \int_0^L u_x \chi_x dx + b \int_0^L (\phi \chi_x + u_x \zeta) dx + \delta \int_0^L \phi_x \zeta_x dx + \xi \int_0^L \phi \zeta dx,
$$

and

$$
\begin{aligned}
l(V) = \rho \int_0^L (i\lambda f_1 + f_2)\chi dx + J \int_0^L (i\lambda f_4 + f_5)\zeta dx \\
- \mu C_1 \chi(L) \int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_3(y) dy \\
- \delta C_2 \zeta(L) \int_{-\infty}^{+\infty} \frac{\omega(y)}{y^2+\eta+i\lambda} f_6(y) dy + \tilde{C}_1 f_1(L)\chi(L) + \tilde{C}_2 f_4(L)\zeta(L).
\end{aligned}
$$

One can easily see that $L_\lambda$, $a_{(H_0^1(0,L))^2}$ and $l$ are bounded. As well, from the incorporated compactness of $H_L^1(0,L)$ in $L^2(0,L)$ it follows that $H_L^1(0,L) \subset L^2(0,L)$ compactly and densely. Furthermore

$$\begin{aligned}\Re a_{(H_L^1(0,L))^2}(U,U) &= \mu\|u_x\|^2_{L^2(0,L)} + 2b\int_0^L u_x\phi dx + \delta\|\phi_x\|^2_{L^2(0,L)} + \xi\|\phi\|^2_{L^2(0,L)} \\ &= \|U\|^2_{(H_L^1(0,L))^2}.\end{aligned}$$

Thus, $a_{(H_0^1(0,L))^2}$ is coercive. Consequently, by Lemma 5, proving the existence of $U$ solution of (58) reduces to proving that (58) with $l \equiv 0$ has a notrivial solution. Indeed if there exists $U \neq 0$ , such that

$$L_\lambda(U,V) + a_{(H_0^1(0,L))^2}(U,V) = 0, \quad \forall V \in H_0^1(0,L)\times H_0^1(0,L).$$

Then $i\lambda$ is an eigenvalue of $\mathcal{A}$.

Therefore, from Lemma 6 we deduce that $U = 0$. □

*Proof of Theorem 4.* Using Lemma 6, we have that $\mathcal{A}$ has no pure imaginary eigenvalues. According to Lemmas 7 and 8 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A})\cap i\mathbb{R} = \{0\}$ if $\eta = 0$. Thus, we get the conclusion by applying Theorem 3 of Arendt and Batty.

The proof of the theorem is complete. □

## 6. Polynomial stability ($\eta \neq 0$)

We will use the following result due to Borichev and Tomilov.

**Theorem 5** ([6], Theorem 2.4)**.** *Let $S(t) = e^{\mathcal{A}t}$ be a bounded $C_0-$semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\mathbb{R} \subseteq \rho(\mathcal{A})$. Then, for a fixed $\beta > 0$ the following conditions are equivalent:*

$$\|(i\lambda I - \mathcal{A})\|_{\mathcal{H}} \leq C|\lambda|^\beta, \quad |\lambda| \to \infty.$$

$$\|S(t)\mathcal{A}^{-1}x\|_{\mathcal{H}} \leq \frac{C}{t^{-\frac{1}{\beta}}}, \quad t \to \infty,\, x \in \mathcal{H}.$$

Our main result is the polynomial stability, given by the following theorem.

**Theorem 6.** *For $U_0 \in D(\mathcal{A})$, the $C_0-$semigroup $S(t) = e^{\mathcal{A}t}$ is polynomially stable, that is,*

$$\|S(t)\mathcal{A}^{-1}U_0\|_{\mathcal{H}} \leqslant \frac{1}{t^{\frac{1}{2(1-\alpha)}}}\|U_0\|_{D(\mathcal{A})}, \quad t > 0.$$

*Proof.* Let's examine the resolvent equation, $(i\lambda I - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, where $U = (u, v, \varphi_1, \phi, \psi, \varphi_2)^T \in D(\mathcal{A})$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, viz

$$\left\{\begin{aligned} i\lambda u - v &= f_1, \\ i\lambda v - \frac{\mu}{\rho} u_{xx} - \frac{b}{\rho}\phi_x &= f_2, \\ i\lambda\varphi_1 + (y^2+\eta)\varphi_1 - v(L)\omega(y) &= f_3, \\ i\lambda\phi - \psi &= f_4, \\ i\lambda\psi - \frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi &= f_5, \\ i\lambda\varphi_2 + (y^2+\eta)\varphi_2 - \psi(L)\omega(y) &= f_6. \end{aligned}\right. \tag{59}$$

Taking the inner product in $\mathcal{H}$ of the resolvent equation with $U$ gives

$$|Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leqslant \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Using (24), comes

$$\mu C_1 \int_{-\infty}^{+\infty} (y^2+\eta)\varphi_1^2 dy + \delta C_2 \int_{-\infty}^{+\infty} (y^2+\eta)\varphi_2^2(y) dy \leqslant \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{60}$$

or

$$\mu C_1 \int_{-\infty}^{+\infty} \varphi_1^2 dy + \delta C_2 \int_{-\infty}^{+\infty} \varphi_2^2(y) dy \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{61}$$

and, applying the second triangular inequality to $(59)_1$ and $(59)_4$, we have

$$\Big||\lambda||u(L)| - |f_1(L)|\Big|^2 \leqslant |v(L)|^2,$$

$$\Big||\lambda||\phi(L)| - |f_4(L)|\Big|^2 \leqslant |\psi(L)|^2,$$

from where we obtain

$$|\lambda|^2|u(L)|^2 \leqslant C|f_1(L)|^2 + C|v(L)|^2, \tag{62}$$

$$|\lambda|^2|\phi(L)|^2 \leqslant C|f_4(L)|^2 + C|\psi(L)|^2, \tag{63}$$

and from $(59)_3$, we get

$$v(L)\omega(y) = (i\lambda + y^2 + \eta)\varphi_1 - f_3(y).$$

Multiplying above equation by $(i\lambda + y^2+\eta)^{-1}\omega(y)$ gives

$$(i\lambda + y^2+\eta)^{-1} v(L)\omega^2(y) = \omega(y)\varphi_1 - (i\lambda + y^2+\eta)^{-1}\omega(y) f_3(y).$$

Now we take the absolute values of both sides of the previous equation, we integrate over the interval $]-\infty,+\infty[$ with respect to the variable $y$ and we apply the Cauchy-Schwarz inequality to obtain

$$S|v(L)| \leqslant U\left(\int_{-\infty}^{+\infty}(y^2+\eta)|\varphi_1|^2dy\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty}|f_3(y)|^2dy\right)^{\frac{1}{2}},$$

where

$$S = \int_{-\infty}^{+\infty}(|\lambda|+y^2+\eta)^{-1}|\omega(y)|^2dy,$$

$$U = \left(\int_{-\infty}^{+\infty}(y^2+\eta)^{-1}|\omega(y)|^2dy\right)^{\frac{1}{2}},$$

$$V = \left(\int_{-\infty}^{+\infty}(|\lambda|+y^2+\eta)^{-2}|\omega(y)|^2dy\right)^{\frac{1}{2}}.$$

Thus, using again the inequality $2PQ \leqslant P^2+Q^2$, $P \geqslant 0$, $Q \geqslant 0$,

$$S^2|v(L)|^2 \leqslant 2U^2\left(\int_{-\infty}^{+\infty}(y^2+\eta)|\varphi_1|^2dy\right) + 2V^2\left(\int_{-\infty}^{+\infty}|f_3(y)|^2dy\right).$$

We deduce that

$$|v(L)|^2 \leqslant C|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}. \tag{64}$$

Similarly to the construction made above, considering $(59)_6$, we have

$$|\psi(L)|^2 \leqslant C|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}, \tag{65}$$

and using the inequalities (64)–(65) in the inequalities (62) and (63), comes

$$|u(L)|^2 \leqslant \frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}}, \tag{66}$$

$$|\phi(L)|^2 \leqslant \frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}}. \tag{67}$$

In addition, by the boundary conditions and the inequality (60) give us

$$|u_x(L)|^2 \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{68}$$

$$|\phi_x(L)|^2 \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{69}$$

Now, we shall introduce the following notations

$$\mathcal{I}_u(\alpha) = \rho|v(\alpha)|^2 + \mu|u_x(\alpha)|^2,$$
$$\mathcal{I}_\phi(\alpha) = J|\psi(\alpha)|^2 + \delta|\phi_x(\alpha)|^2,$$
$$\mathcal{I}(\alpha) = \mathcal{I}_u(\alpha) + \mathcal{I}_\phi(\alpha),$$

$$\varepsilon_u(L) = \int_0^L \mathcal{I}_u(s)ds, \quad \varepsilon_\phi(L) = \int_0^L \mathcal{I}_\phi(s)ds.$$

**Lemma 9.** *Let $q \in H^1(0,L)$. Then we have*

(70) $\varepsilon_u(L) = [q\mathcal{I}_u]_0^L + 2bRe\int_0^L q\phi_x\overline{u_x}dx + R_1,$

(71) $\varepsilon_\phi(L) = [q\mathcal{I}_\phi]_0^L - \xi\left[q|\phi|^2\right]_0^L - 2bRe\int_0^L qu_x\overline{\phi_x}dx + \int_0^L q'|\phi|^2dx + R_2,$

*where $R_i$ satisfy*

$$|R_i| \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \quad \forall\, i = 1,2,$$

*for a positive constant $C$.*

*Proof.* To get (70), let's multiply the equation $(59)_2$ by $q\overline{u_x}$ and integrate in $(0,L)$, which gives us

$$i\lambda\rho\int_0^L vq\overline{u_x}dx - \mu\int_0^L u_{xx}q\overline{u_x}dx - b\int_0^L \phi_x q\overline{u_x}dx = \rho\int_0^L f_2q\overline{u_x}dx,$$

or

$$-\rho\int_0^L vq\overline{i\lambda u_x}dx - \mu\int_0^L qu_{xx}\overline{u_x}dx - b\int_0^L q\phi_x\overline{u_x}dx = \rho\int_0^L f_2q\overline{u_x}dx.$$

From $(59)_1$ we have that $i\lambda u_x = v_x + f_{1x}$, then taking the real part, the above equality leads to

$$\begin{aligned}&-\frac{\rho}{2}\int_0^L q\frac{d}{dx}|v|^2dx - \frac{\mu}{2}\int_0^L q\frac{d}{dx}(|u_x|^2)dx\\ &= \rho Re\int_0^L f_2q\overline{u_x}dx + \rho Re\int_0^L vq\overline{f_{1x}}dx + bRe\int_0^L q\phi_x\overline{u_x}dx.\end{aligned}$$

Integrating by parts, we have

$$\int_0^L q'(s)\left[|v(s)|^2 + \mu|u_x(s)|^2\right]ds = [q\mathcal{I}_u]_0^L + 2bRe\int_0^L q\phi_x\overline{u_x}dx + R_1,$$

where

$$R_1 = 2\rho Re\int_0^L f_2q\overline{u_x}dx + 2\rho Re\int_0^L vq\overline{f_{1x}}dx,$$

in this way we can obtain the equality (70) with

$$|R_1| \leqslant 2\rho\int_0^L |q||f_2||u_x|dx + 2\rho\int_0^L |q||v||f_{1x}|dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$

which proves the first part of the Lemma 9.

Similarly, if we multiply $(59)_5$ by $q\overline{\phi_x}$ and integrate in $(0,L)$, we get

$$\begin{aligned}&i\lambda J\int_0^L \psi q\overline{\phi_x}dx - \delta\int_0^L \phi_{xx}q\overline{\phi_x}dx +\\ &\quad + b\int_0^L u_xq\overline{\phi_x}dx + \xi\int_0^L \phi q\overline{\phi_x}dx = J\int_0^L f_5q\overline{\phi_x}dx,\end{aligned}$$

or

$$-J\int_0^L \psi q(\overline{i\lambda\phi_x})dx - \delta\int_0^L q\phi_{xx}\overline{\phi_x}dx +$$
$$+b\int_0^L qu_x\overline{\phi_x}dx + \xi\int_0^L q\phi\overline{\phi_x}dx = J\int_0^L qf_5\overline{\phi_x}dx,$$

From $(59)_4$, it follows that $i\lambda\phi_x = \psi_x + f_{4x}$, then taking the real part, the above equality results in

$$-\frac{J}{2}\int_0^L q\frac{d}{dx}|\psi|^2dx - \frac{\delta}{2}\int_0^L q\frac{1}{2}\frac{d}{dx}|\phi_x|^2dx + \frac{\xi}{2}\int_0^L q\frac{d}{dx}|\phi|^2dx$$
$$= JRe\int_0^L qf_5\overline{\phi_x}dx + JRe\int_0^L q\psi\overline{f_{4x}}dx - bRe\int_0^L qu_x\overline{\phi_x}dx.$$

Integrating by parts, we obtain

$$\int_0^L q'(s)\left[J|\psi(s)|^2 + \delta|\phi_x(s)|^2\right]ds$$
$$= [q\mathcal{I}_\phi]_0^L - \xi\left[q|\phi|^2\right]_0^L - 2bRe\int_0^L qu_x\overline{\phi_x}dx + \xi\int_0^L q'|\phi|^2dx + R_2,$$

where

$$R_2 = JRe\int_0^L qf_5\overline{\phi_x}dx + JRe\int_0^L q\psi\overline{f_{4x}}dx.$$

This gives us the equality (71) with

$$|R_2| \leqslant 2J\int_0^L |q||f_5||\phi_x|dx + 2J\int_0^L |q||\psi||f_{4x}|dx \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \quad \square$$

Taking $q:[0,L]\longrightarrow[0,L]\subset\mathbb{R}$ given by $q(x)=x$ in Lemma 9 and then adding (70)–(71), we get

$$\varepsilon_u(L) + \varepsilon_\phi(L) = L\left(\mathcal{I}_u(L) + \mathcal{I}_\phi(L)\right) - \xi L|\phi(L)|^2$$
$$+\xi\int_0^L |\phi|^2dx + R_1 + R_2,$$

as $|R_i| \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$, $\forall\, i=1,2$, we have

$$\varepsilon_u(L) + \varepsilon_\phi(L) \leqslant L\left(\mathcal{I}_u(L) + \mathcal{I}_\phi(L)\right) + L\xi|\phi(L)|^2 + \xi\int_0^L |\phi|^2dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$$
$$= L\left(\rho\underbrace{|v(L)|^2}_{(64)} + \mu\underbrace{|u_x(L)|^2}_{(68)} + J\underbrace{|\psi(L)|^2}_{(65)} + \delta\underbrace{|\phi_x(L)|^2}_{(69)}\right)$$
$$+ L\xi\underbrace{|\phi(L)|^2}_{(67)} + \xi\int_0^L |\phi|^2dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}$$

$$\leqslant \xi \int_0^L |\phi|^2 dx + C\left(|\lambda|^{2-2\alpha}+1\right)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}, \tag{72}$$

for $\lambda \neq 0$. On the other hand, from $(59)_4$ it follows that

$$\phi = \frac{\psi + f_4}{i\lambda} \quad \Rightarrow \quad \int_0^L |\phi|^2 dx \leqslant \frac{C}{|\lambda|^2}\left(\|U\|^2_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right). \tag{73}$$

Then, if we use (73) in the inequality (72), we get

$$\varepsilon_u(L) + \varepsilon_\phi(L) \leqslant C\left(|\lambda|^{2-2\alpha}+1\right)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}. \tag{74}$$

Now, we multiply $(59)_5$ by $\overline{\phi}$ and integrate in $(0, L)$, in order to obtain

$$i\lambda J \int_0^L \psi\overline{\phi}dx - \delta \int_0^L \phi_{xx}\overline{\phi}dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L \phi\overline{\phi}dx = J\int_0^L f_5\overline{\phi}dx. \tag{75}$$

Combining $(59)_4$ and (75), it results

$$\begin{aligned} &-\delta \int_0^L \phi_{xx}\overline{\phi}dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2 dx \\ &= J\int_0^L |\psi|^2 dx + J\int_0^L f_5\overline{\phi}dx + J\int_0^L \psi\overline{f_4}dx. \end{aligned}$$

Integrating by parts and using the boundary conditions, we have

$$\begin{aligned} &\delta \int_0^L |\phi_x|^2 dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2 dx \\ &= J\int_0^L |\psi|^2 dx + J\int_0^L (\psi\overline{f_4} + f_5\overline{\phi})dx + \overline{\phi}(L)\phi_x(L) \\ &\leqslant J\int_0^L |\psi|^2 dx + J\int_0^L (|\psi||f_4| + |f_5||\phi|)dx + |\phi(L)||\phi_x(L)|, \end{aligned}$$

using Young's inequality, comes

$$\begin{aligned} &\delta \int_0^L |\phi_x|^2 dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2 dx \\ &\leqslant C\varepsilon_\phi(L) + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{1}{2}|\phi(L)|^2 + \frac{1}{2}|\phi_x(L)|^2. \end{aligned}$$

Then, using the inequalities (67) and (69) it follows that

$$\begin{aligned}&\delta\int_0^L|\phi_x|^2dx+b\int_0^L u_x\overline{\phi}dx+\xi\int_0^L|\phi|^2dx\\ &\leqslant C\varepsilon_\phi(L)+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.\end{aligned}\tag{76}$$

Finally, we multiply $(59)_2$ by $\overline{u}$, and we integrate in $(0,L)$, obtaining

$$i\lambda\rho\int_0^L v\overline{u}dx-\mu\int_0^L u_{xx}\overline{u}dx-b\int_0^L\phi_x\overline{u}dx=\rho\int_0^L f_2\overline{u}dx,$$

that is

$$-\rho\int_0^L v\overline{i\lambda u}dx-\mu\int_0^L u_{xx}\overline{u}dx-b\int_0^L\phi_x\overline{u}dx=\rho\int_0^L f_2\overline{u}dx.$$

From $(59)_1$, we get

$$-\mu\int_0^L u_{xx}\overline{u}dx-b\int_0^L\phi_x\overline{u}dx=\rho\int_0^L|v|^2dx+\rho\int_0^L f_2\overline{u}dx+\rho\int_0^L v\overline{f_1}dx,$$

from where, we obtain

$$\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx=\rho\int_0^L|v|^2dx+\rho\int_0^L(v\overline{f_1}+f_2\overline{u})dx+\mu\overline{u}(L)u_x(L),$$

thus

$$\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx\leqslant C\varepsilon_u(L)+\rho\int_0^L(|v||f_1|+|f_2||u|)dx+\mu|u(L)||u_x(L)|.$$

Young's inequality gives

$$\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx\leqslant C\varepsilon_u(L)+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{\mu^2}{2}|u(L)|^2+\frac{1}{2}|u_x(L)|^2,$$

and from inequalities (66) and (68), we obtain

$$\begin{aligned}&\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx\\ &\leqslant C\varepsilon_u(L)+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.\end{aligned}\tag{77}$$

Adding the inequalities (76) and (77), we find

$$\begin{aligned}&\int_0^L|u_x|^2dx+b\int_0^L(\phi\overline{u_x}+u_x\overline{\phi})dx+\xi\int_0^L|\phi|^2dx+\delta\int_0^L|\phi_x|^2dx\\ &\leqslant C\left(\varepsilon_u(L)+\varepsilon_\phi(L)\right)+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.\end{aligned}\tag{78}$$

Therefore, from (61), (74) and (78), we conclude that

$$\|U\|^2_{\mathcal{H}}\leqslant C\left(\varepsilon_u(L)+\varepsilon_\phi(L)\right)+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|F\|^2_{\mathcal{H}},$$

that is

$$\|U\|_{\mathcal{H}}^2 \leqslant C\left(|\lambda|^{2-2\alpha}+1\right)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|U\|_{\mathcal{H}}^2$$
$$+\frac{C}{|\lambda|^{2\alpha}}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\|F\|_{\mathcal{H}}^2+C\|F\|_{\mathcal{H}}^2.$$

For $\lambda \neq 0$ and applying Young's inequality, we get the following result

$$\|U\|_{\mathcal{H}}^2 \leqslant C|\lambda|^{4(1-\alpha)}\|F\|_{\mathcal{H}}^2,$$

that is

$$\|U\|_{\mathcal{H}} \leqslant C|\lambda|^{2(1-\alpha)}\|F\|_{\mathcal{H}}, \quad \forall U \in D(\mathcal{A}),$$

which is equivalent to

$$\frac{\|(i\lambda I-\mathcal{A})^{-1}F\|_{\mathcal{H}}}{\|F\|_{\mathcal{H}}} \leqslant C|\lambda|^{2(1-\alpha)} \quad \Rightarrow \quad \|(i\lambda I-\mathcal{A})^{-1}\|_{\mathcal{H}} \leqslant C|\lambda|^{2(1-\alpha)}.$$

for some positive constant $C$. The conclusion follows from Theorem 5. □

## 7. Conclusions

Properties of a one-dimensional poroelastic system, such as the existence and uniqueness of solution, strong stability, and polynomial stability, were analyzed. It was shown that the $C_0$-semigroup associated with the system (4)–(7), although without internal dissipation, presents strong stability due to the presence of two dissipative mechanisms at the boundary, based on fractional derivatives. Furthermore, it was proven that the system decays polynomially with a rate of $t^{-1/2(1-\alpha)}$, as seen in Sections 5 and 6, respectively. Some issues may be considered, such as: In Theorem 6, the decay rate of order $t^{-1/2(1-\alpha)}$ is obtained. It is interesting to prove that this decay rate is optimal by the Borichev-Tomilov theorem. The decay rate in the case $\eta = 0$ is an open question. Since $\lambda = 0$ is a spectral value, the Borichev-Tomilov theorem does not apply. Other methods can be tested, in particular, observability theory. Another technique is the Laplace transform and the representation of solutions by Mittag-Leffler functions.

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